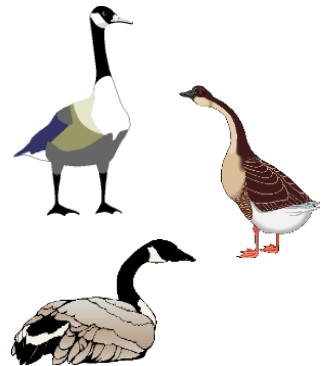

SETS



A *bunch* of bananas, a *gaggle* of geese — each is an example of a **set**, referred to as an *ensemble* in French. The idea of a set is so fundamental that it doesn't even have a definition, yet sets form the foundation of all mathematics.



□ BASIC SET DEFINITIONS

A **set** is a well-defined collection of objects (but what's a collection?). Well-defined means that, for instance, “all whole numbers greater than 10” is a set, but “all tall people” is not a set (it's too vague).

Consider the sets

$$A = \{1, 3, 5\} \quad B = \{1, 3, 5, 7, 10\}$$

The **elements** (or members) of set A are 1, 3, and 5, surrounded by (curly) braces. We say “3 is an element of A” and write “ $3 \in A$ ”. We can also say that “12 is not an element of A”, in which case we write “ $12 \notin A$ ”.

In the sets A and B above, notice that all the elements of A are also elements of B. In a sense, A is contained in B. We say that “A is a **subset** of B” and write “ $A \subseteq B$ ”. Can you see that B is not a subset of A? We write “ $B \not\subseteq A$ ”.

There's a very special set that contains *no* elements. It is called the **null set** (or the empty set), and is written $\emptyset$. Some teachers prefer to write $\{ \}$ for the null set.

Most books use the symbol $\mathbb{N}$ to denote the set of all **natural numbers**:

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

Also, we'll use the symbol $\mathbb{Z}$ to represent the set of *integers*:

$$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$$

Z is the first letter of **Zahl**, the German word for *number*.

Homework

1. Consider the collection of all registered students in this class. Is this a set? Why or why not?
2. Consider the collection of all students at the college with a high GPA. Is this a set? Why or why not?
3. Fill in the blanks:
 - a. 7 ____ { all primes }
 - b. 13 ____ { all even numbers }
4. Fill in the blanks:
 - a. 6 ____ {3, 6, 9}
 - b. {6} ____ {3, 6, 9}
5.
 - a. T/F: {students in this class} $\subseteq$ { DVC students }
 - b. T/F: {people in this class} $\subseteq$ { DVC students }
6. T/F: {primes} $\subseteq$ {odd natural numbers}
Hint: The first few primes are 2, 3, 5, 7, 11, 13,
7. Explain why $\{7, 8, 9\} \not\subseteq \{7, 9, 11, 13, 20\}$.
8. T/F: $\{\emptyset\} = \emptyset$. Be sure to explain your answer.
9. Which statement is true, $\{1, 9, 21\} \subseteq \mathbb{N}$ or $\mathbb{N} \subseteq \{1, 9, 21\}$?
10. Prove that $\mathbb{N} \subseteq \mathbb{Z}$. Now prove that $\mathbb{Z} \not\subseteq \mathbb{N}$.
11. Fill in the blank: If $A \subseteq B$ and $B \subseteq A$, then _____.
12. T/F: $\{1, 3\} \in \{\{a\}, \{x, y\}, \{1, 3\}\}$

Do you see that
 $\mathbb{N} \subseteq \mathbb{Z}$?

❑ **IMPORTANT PROPERTIES OF SETS**

- ◆ The elements of a set may be explicitly listed, such as the set of positive even numbers: $\{2, 4, 6, 8, \dots\}$. The positive even numbers can also be written $\{n \in \mathbb{N} \mid n \text{ is even}\}$, where the vertical bar, $\mid$, is read “*such that*.” [Some books write this with a colon instead of a vertical bar: $\{n \in \mathbb{N}: n \text{ is even}\}$]
- ◆ The order in which the elements of a set are listed doesn’t matter. For example, $\{1, 2, 3\} = \{2, 3, 1\}$.
- ◆ Repeated elements in a set can be thrown out without changing the set. For instance, $\{a, b, a, c\} = \{a, b, c\}$.
- ◆ Is every element of the set $\{1, 2, 3\}$ also in the set $\{1, 2, 3\}$? Yes, so $\{1, 2, 3\} \subseteq \{1, 2, 3\}$. In general, for any set A ,

$$A \subseteq A$$

- ◆ We now contend that $\emptyset \subseteq \{1, 2, 3\}$. This is really strange, but look at it this way: If the null set were not a subset of $\{1, 2, 3\}$, then there would have to be something in the null set that fails to be in $\{1, 2, 3\}$. But this is impossible since there’s nothing in the null set! Thus, it can not be the case that the null set is not a subset of $\{1, 2, 3\}$, and therefore it is a subset: $\emptyset \subseteq \{1, 2, 3\}$. This may be hard to fathom, so you may have to accept it on faith. Just remember the following fact: If A is any set,

$$\emptyset \subseteq A$$

Homework

13. List the elements of $\{x \in \mathbb{N} \mid x \leq 5\}$.
14. List the elements of $\{x \in \mathbb{N} \mid 5 \leq x < 10\}$.
15. Write the set $\{2, 4, 6, 8, 10\}$ in the form $\{x \in \mathbb{N} \mid x \text{ is } \dots\}$.
16. Write the set $\{2, 3, 5, 7, 11, 13\}$ in the form $\{x \in \mathbb{N} \mid x \text{ is } \dots\}$.
17. T/F: $\{6, 7, 8, 7, 4\} = \{4, 7, 6, 8\}$. Explain.
18. T/F: $\{a, b, c, d, e\} = \{b, c, d, e\}$. Explain.
19. T/F: $\{1, 5, 9\} \subseteq \{1, 5, 9\}$
20. T/F: $\emptyset \subseteq \emptyset$. Explain.
21. The number of elements in a set is called its **cardinality**. For example, the cardinality of $\{2, 4, 8, 16, 32\}$ is 5. What is the cardinality of $\emptyset$? What is the cardinality of $\{1, 3, 5, 7, \dots, 21\}$?
22. Let A be a set with cardinality 100. What is the cardinality of $\{A\}$? [This is tricky.]
23. What is the cardinality of $\mathbb{N}$? What is the cardinality of the set of even numbers? Do $\mathbb{N}$ and the even numbers have the same cardinality?

□ **COMBINING SETS**

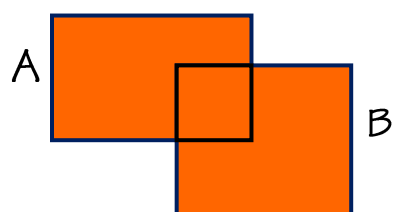
The **union** of two sets A and B is the new set obtained by lumping together all the elements that are in A or B (or both). The union of A and B is written $\mathbf{A} \cup \mathbf{B}$. For example,

$$\{1, 2, 7\} \cup \{2, x, \Delta\} = \{1, 2, 7, x, \Delta\}$$

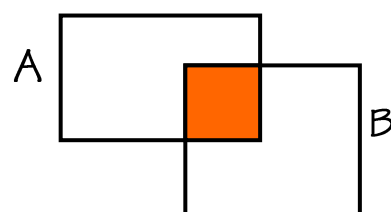
The ***intersection*** of sets A and B is the new set obtained by extracting only the elements that are in both A and B — that is, the elements that A and B have *in common*. The intersection of A and B is written $\mathbf{A} \cap \mathbf{B}$. For example,

$$\{a, b, R\} \cap \{b, c, R\} = \{b, R\}$$

Union and intersection can be displayed as *Venn diagrams*:



$A \cup B$ is shaded



$A \cap B$ is shaded

The following are the official definitions of union and intersection:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

EXAMPLE 1:

- A. $\{a, b, c\} \cup \{d, w, z\} = \{a, b, c, d, w, z\}$
- B. $\{1, 3, 5, 7\} \cap \{2, 4, 6\} = \emptyset$
- C. $\{m, n, p\} \cup \{m, n, o, p, q\} = \{m, n, o, p, q\}$
- D. $\{7, 8, 9\} \cap \{7, 8, 9, 10\} = \{7, 8, 9\}$
- E. $\{a, 1, 2, \pi\} \cup \emptyset = \{a, 1, 2, \pi\}$
- F. $\{x, y, z\} \cap \emptyset = \emptyset$

Homework

24. Sketch a Venn diagram representing the statement: $A \subseteq B$.
25. Use a Venn diagram to prove:
 If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
26. Find $\{a, b, c\} \cup \{b, d, w, z\} \cup \{a, d, w, m, z\}$.
27. Find $\{1, 3, 5\} \cap \{3, 5, 7\} \cap \{5, 7, 9\}$.
28. Two sets A and B are said to be **disjoint** if $A \cap B = \emptyset$; for example, the set of even numbers and the set of odd numbers are disjoint. Sketch a Venn diagram of two disjoint sets.
29. Find a *counterexample* to the conjecture:
 $A \cup (B \cap C) = (A \cup B) \cap C$
 Hint: Find sets A , B , and C that make the statement false.
30. Use the sets $A = \{1, 2, 3\}$, $B = \{2, 3, 7\}$, and $C = \{3, 7, 9\}$ to give some evidence for a *distributive property* of sets:
 $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 Does this example constitute a proof of a distributive property of sets?

❑ FINDING ALL THE SUBSETS OF A SET

EXAMPLE 2: List all the subsets of $\{1, 2, 3\}$.

Solution: Recall the Important Properties section a few pages back. There, we learned that the null set, $\emptyset$, is a subset of any set; thus $\emptyset$ is one of the subsets of $\{1, 2, 3\}$. Also, we discussed

the fact that any set is a subset of itself — thus, $\{1, 2, 3\}$ is one of the subsets of $\{1, 2, 3\}$. That’s two subsets so far — essentially the “smallest” and the “largest” subsets.

Next, we’ll list the subsets containing exactly one element:

$$\{1\} \quad \{2\} \quad \{3\}$$

And finally, we’ll list the subsets containing exactly two elements:

$$\{1, 2\} \quad \{1, 3\} \quad \{2, 3\}$$

Remembering the null set and the entire set, we obtain a total of eight subsets:

$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$

EXAMPLE 3: List all the subsets of $\{a, b, c, d\}$.

Solution: We’ll organize the subsets in order of *cardinality* (the number of elements in the subset):

0 elements:	$\emptyset$
1 element:	$\{a\} \quad \{b\} \quad \{c\} \quad \{d\}$
2 elements:	$\{a, b\} \quad \{a, c\} \quad \{a, d\} \quad \{b, c\} \quad \{b, d\} \quad \{c, d\}$
3 elements:	$\{a, b, c\} \quad \{a, b, d\} \quad \{a, c, d\} \quad \{b, c, d\}$
4 elements:	$\{a, b, c, d\}$

Count ‘em all up — there should be **16 subsets** altogether.

Homework

31. a. List all the subsets of $\{a, b\}$.
 b. List all the subsets of $\{x\}$.
 c. List all the subsets of $\emptyset$.
32. How many subsets does the set $\{a, b, c, d, e, f, g, h, i, j\}$ have?
33. Suppose a set has cardinality n . How many subsets does the set have?
34. How many subsets does the set $\{\{1\}, \{1, 2\}, \{7, 9, 10\}\}$ have?
35. A set X has 128 subsets. Find the cardinality of X .

Review Problems

36. What's the distinction between a set and a non-set?
37. a. T/F: $\{5\} = 5$
 b. $A \cup A = \underline{\hspace{2cm}}$
 c. $A \cap A = \underline{\hspace{2cm}}$
38. We know that $\mathbb{N}$ has cardinality ∞ . What is the cardinality of $\{\mathbb{N}\}$?
39. T/F: $\{1, 3\} \in \{0, 1, 2, 3, 4, 5\}$
40. Explicitly list the elements of the set $A = \{x \in \mathbb{N} \mid x \text{ is even and } x < 7\}$.

41. T/F: If $A \subseteq B$ and $C \subseteq B$, then $A \cup C = B$.
42. For any set A ,
 $A \cap \emptyset = \underline{\hspace{2cm}}$ and $A \cup \emptyset = \underline{\hspace{2cm}}$.
43. If $A = \{4, 5, 9\}$, find a set B such that $A \cap B = \{5, 9\}$.
44. How many subsets does the set $\{5, 12, 3, a, 7, x, \frac{3}{8}\}$ have?
45. a. How many subsets does $\emptyset$ have?
b. How many subsets does $\{\emptyset\}$ have?
46. Find the cardinality of a set which has 512 subsets.
47. T/F: $\{5, 6\} = \{6, 5\}$
48. The set C has cardinality 99. What is the cardinality of $\{C\}$?
49. T/F: $\{1, 3\} \subseteq \{0, 1, 2, 3, 4, 5\}$
50. Explicitly list the elements of the set $A = \{x \in \mathbb{N} \mid x \text{ is odd and } x < 10\}$.
51. T/F: If $A \subseteq B$ and $B \subseteq C$, then $C \subseteq A$.
52. For any set A ,
 $A \cap A = \underline{\hspace{2cm}}$ and $A \cup A = \underline{\hspace{2cm}}$.
53. If $A = \{4, 5, 9\}$, find a set B such that $A \cup B = \{1, 4, 5, 9\}$.
54. How many subsets does the set $\{5, 12, 3, a, 7, y, z, w\}$ have?
55. a. How many subsets does $\emptyset$ have?
b. How many subsets does $\{\emptyset, \{\emptyset\}\}$ have?

56. Find the cardinality of a set which has 1024 subsets.
57. T/F: For any sets A and B, $A \cup B = A \cap B$. Prove your answer.
58. True/False:
- $\{1\} \in \{0, 2, \{1\}, \pi\}$
 - If Y is any set, $Y \subseteq Y$.
 - If Z is any set, $\emptyset \in Z$.
 - $\emptyset \subseteq \emptyset$.
 - $\{a, b, c\} \cup \{b, c, d\} = \{c\}$.
 - $\{1, 2\} \cap \{3, 9\} = \emptyset$.
 - If A is any set, $A \cup \emptyset = \emptyset$.
 - $\{1, 2\}$ and $\{3, \pi\}$ are disjoint.
 - If B is any set, B and $\emptyset$ are disjoint.
 - A set has 5 elements. It follows that the set has 25 subsets.
 - A set has 2048 subsets. It follows that the set has 11 elements.
 - If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
 - If $A \subseteq B$, then $A \cup B = B$.
 - If $A \subseteq B$, then $A \cap B = A$.

□ TO ∞ AND BEYOND

Let A be any set. We will denote the **cardinality** of A (the number of elements in A) by the notation $n(A)$. We also define $\mathcal{P}(A)$, called the **power set** of A, to be the set of all subsets of A.

For example, let $A = \{1, 5\}$. Then

$$n(A) = 2 \text{ and } \mathcal{P}(A) = \{\emptyset, \{1\}, \{5\}, \{1, 5\}\}$$

Can you see that $n(\mathcal{P}(A)) = 4$?

A. State the definition of $\mathcal{P}(A)$ (fill in the blank):

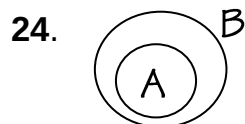
$$\mathcal{P}(A) = \{ X \mid \underline{\hspace{2cm}} \}$$

B. If A is a set such that $n(A) = 12$, calculate $n(\mathcal{P}(A))$.

C. Suppose B is a set such that $n(\mathcal{P}(B)) = 256$. Calculate $n(B)$.

Solutions

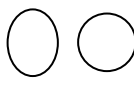
1. Yes; well-defined. 2. No; exactly what is a high GPA?
3. $\in, \notin$ 4. $\in, \subseteq$ 5. a. T b. F 6. F
7. 8 is an element of the first set, but it's not an element of the second set.
8. F — one set has 1 element in it, while the other set has 0 elements in it.
9. The first 10. Every natural number is an integer, so $\mathbb{N} \subseteq \mathbb{Z}$. But $-3 \in \mathbb{Z}$, yet $-3 \notin \mathbb{N}$; thus $\mathbb{Z} \not\subseteq \mathbb{N}$.
11. $A = B$ 12. T 13. $\{1, 2, 3, 4, 5\}$ 14. $\{5, 6, 7, 8, 9\}$
15. $\{x \in \mathbb{N} \mid x \text{ is even and } x \leq 10\}$ 16. $\{x \in \mathbb{N} \mid x \text{ is a prime number } \leq 13\}$
17. T — duplicates can be discarded, and the order doesn't matter.
18. F 19. T 20. T — the null set is a subset of every set, including itself. 21. 0; 11 22. 1
23. $\infty; \infty$; What do you think?



25. Draw A inside B and B inside C. Is it clear that A is inside C, and therefore that $A \subseteq C$?

26. $\{a, b, c, d, w, z, m\}$

27. $\{5\}$

28. 
A B

29. Let me know what sets you came up with.

30. Work out each side by doing what's in parentheses first. An example does not prove anything.

31. a. $\{a, b\}, \{a\}, \{b\}, \emptyset$ b. $\{x\}, \emptyset$ c. $\emptyset$

32. 1024

33. What do you think?

34. 8

35. 7

36. A set is well-defined; it's always possible to tell whether something is in the set or not.

37. a. False b. A c. A 38. 1 39. False 40. $A = \{2, 4, 6\}$

41. False 42. $\emptyset; A$ 43. $B = \{5, 7, 9\}$, for example

44. 128 45. a. 1 b. 2 46. 9

47. a. T 48. 1 49. T

50. $A = \{1, 3, 5, 7, 9\}$ 51. F 52. A; A 53. $B = \{1\}$

54. 256 55. a. 1 b. 4 56. 10

57. False; for example, let $A = \{1, 2\}$ and $B = \{3\}$. Then
 $A \cup B = \{1, 2, 3\}$, while $A \cap B = \emptyset$.

58. a. T b. T c. F d. T e. F f. T g. F h. T
i. T j. F k. T l. T m. T n. T

*“Our progress as a nation can
be no swifter than our progress
in education.”*

– John Fitzgerald Kennedy